$$B = h\left(\frac{m}{Q}\right)^2 \frac{2\alpha(\alpha+1)}{(\alpha-1)^2} y_1^3 \left[-\frac{16g}{l}\frac{\alpha^2-4}{9\alpha^2-4}\left(\frac{mh}{Q}\right)^2 y_2\right]^b.$$

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DYNAMIC DEFORMATION OF A WEDGE MADE OF AN INHOMOGENEOUS HARDENING MATERIAL

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UDC 539.374

Consideration is given to compression and bending of a plane infinite wedge at whose tip a concentrated force P(t) is applied varying with time by a special rule. The material is assumed to be incompressible, plastically inhomogeneous, and it obeys an exponential hardening rule. In essence, this material may also relate to a nonlinearly elastic, nonlinearly ductile body whose compressibility may ignored. A study is also made of the effect of external forces with which points of the body complete vibratory and monotonic movements in time. Concentrated forces are determined corresponding to the deformed state of the wedge being considered. Questions of unloading are not discussed, and therefore for the case of plastic bodies a study is also made of the stages of movement which lead to loading.

The stressed state in plastically inhomogeneous bodies under dynamic effects has been studied in [1-3, and others]. A similar analysis of dynamic problems for plastically inhomogeneous bodies is given in [4, 5]. A study of dynamic deformation questions for a plastically inhomogeneous incompressible body is of interest, particularly from the point of view of studying the effect of inertial forces on the stressed–strained state of the body.

Dynamic problems for incompressible ductile materials with axisymmetric and planar deformation have been considered in [6, 7].

1. Equations for deformation theory of plasticity for an incompressible, inhomogeneous material with an exponential hardening rule in the case of plane strain have in normal notation the following form:

differential equations of motion

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} = \rho\frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2}{r}\tau_{r\theta} = \rho\frac{\partial^2 v}{\partial t^2}; \tag{1.1}$$

relationships between stress and strain intensities

$$\varepsilon_0 = \left(\frac{\sigma_0}{K}\right)^3, \quad K = K(r,\theta),$$
$$\sigma_0 = \frac{1}{2}\sqrt{(\sigma_r-\sigma_\theta)^2 + 4\tau_{r\theta}^2}, \quad \varepsilon_0 = \sqrt{(\varepsilon_r-\varepsilon_\theta)^2 + 4\gamma_{r\theta}^2} \tag{1.2}$$

[K(r, θ) is a known function characterizing a plastically inhomogeneous material];

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relationships between strain, displacement, and stress components

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{1}{2} k(r, \theta) \sigma_0^2 (\sigma_r - \sigma),$$
$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{2} k(r, \theta) \sigma_0^2 (\sigma_\theta - \sigma), \tag{1.3}$$
$$2\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = k(r, \theta) \sigma_0^2 \tau_{r\theta}$$

[$k(r, \theta) = K^{-3}(r, \theta)$, $\sigma = (1/2)(\sigma_r + \sigma_\theta)$ is the average stress]. Subsequently it is assumed that inhomogeneity is governed by the rule

$$K(r, \theta) = k\omega(\theta), \tag{1.4}$$

where k is a constant value; $\omega(\theta)$ is a known function determined by experiment.

Displacement components satisfying the incompressibility condition are presented as

$$u = r^{-\lambda-1}\psi'(\theta)f(t), \quad v = \lambda r^{-\lambda-1}\psi(\theta)f(t) \tag{1.5}$$

[$\psi(\theta)$ and $f(\theta)$ are arbitrary functions of their own arguments; λ is a constant parameter]. Proceeding from relationship (1.3) for the stress component we have

$$\sigma_r = \sigma_\theta - 4k(\lambda + 1) r^{-\frac{\lambda+2}{3}} \psi'(\theta)\omega(\theta)\chi(\theta) f^{1/3}(t),$$
$$\tau_{r\theta} = kr^{-\frac{\lambda+2}{3}} [\psi''(\theta) - \lambda(\lambda + 2)\psi(\theta)]\omega(\theta)\chi(\theta) f^{1/3}(t), \tag{1.6}$$
$$\chi(\theta) = \{4(\lambda+1)^2\psi'^2(\theta) + [\psi''(\theta) - \lambda(\lambda+2)\psi(\theta)]^2\}^{-\frac{1}{3}}.$$

By substituting expressions for displacements (1.5) and stresses (1.6) in the first equation of (1.1), we obtain

$$\sigma_\theta = H(t) + \frac{3kr^{-\frac{\lambda+2}{3}}}{\lambda+2} f^{1/3}(t)\Big\{([\psi''(\theta) - \lambda(\lambda+2)\psi(\theta)]\omega(\theta)\chi(\theta))' +$$
$$+ \frac{4}{3}(\lambda^2 - 1)\psi'(\theta)\omega(\theta)\chi(\theta)\Big\} - \frac{\rho}{\lambda} r^{-\lambda}\psi'(\theta) f''(t) \tag{1.7}$$

[H(t) is an arbitrary function of the argument]. Expressions (1.5)-(1.7) will be solved by Eqs. (1.1) if $\lambda = 1$, and function $\psi(\theta)$ satisfies a normal second order differential equation

$$\frac{\omega(\psi'' - 3\psi)}{\sqrt[3]{(\psi'' - 3\psi)^2 - 16\psi'^2}} - \nu\psi = A\sin(\theta + \delta). \tag{1.8}$$

Here $\nu = \pm\mu^2\rho/k$; μ is a constant value; A and δ are arbitrary constants. Function f(t) satisfies a second order differential equation

$$\ddot{f} \pm \mu f^{1/3} = 0, \tag{1.9}$$

whose solution with $\mu \neq 0$ is presented in quadratures:

$$\sqrt{\frac{3}{2}}\mu t = \pm\int_{f_0}^{f} dx/\sqrt{c^{4/3} \mp x^{4/3}}, \tag{1.10}$$

where f_0 and c are parameters characterizing dynamic deformation. A minus in the expression under the root corresponds to nonlinear vibration of the body, and a plus to deformation monotonic with time. A curve for relationship (1.10) with $f_0 = 0$ is given in Fig. 1.

Equations for stresses and displacements take the form

$$\sigma_r = \frac{kf^{1/3}(t)}{r}\left[A\cos(\theta + \delta) - \frac{8\psi'\omega}{\sqrt[3]{(\psi'' - 3\psi)^2 + 16\psi'^2}}\right] + H(t), \tag{1.11}$$

$$\sigma_\theta = \frac{kf^{1/3}(t)}{r} A\cos(\theta+\delta) + H(t),\ \tau_{r\theta} = \frac{kf^{1/3}(t)}{r}[A\sin(\theta+\delta)+\nu\psi],$$

$$u = \frac{f(t)}{r^2}\psi'(\theta),\ v = \frac{f(t)}{r^2}\psi(\theta).$$

Proceeding from the condition $\sigma_\theta = 0$ with $\theta = \pm\alpha$, in relationships (1.11) we obtain A = 0, H(t) = 0. Equation (1.8) is written as

$$\frac{\omega(\psi''-3\psi)}{\sqrt[3]{(\psi''-3\psi)^2+16\psi'^2}} - \nu\psi = 0. \tag{1.12}$$

Differential Eq. (1.12) is reduced to a cubic equation relating to $(\psi''-3\psi)^{-1}$. By determining its actual root we arrive at a differential equation

$$\psi''-3\psi = \frac{4\sqrt{3}\,\nu\psi\psi'}{\sqrt[3]{6\sqrt{3}\,\psi'\omega^3+\sqrt{\nu^6\psi^6+108\omega^6\psi'^2}}+\sqrt[3]{6\sqrt{3}\,\psi'\omega^3-\sqrt{\nu^6\psi^6+108\omega^6\psi'^2}}}. \tag{1.13}$$

2. In the case of compressing a wedge by a concentrated force applied to its tip it is necessary to integrate differential Eq. (1.13) with boundary conditions

$$\psi = 0 \text{ for } \theta = 0,\ \psi = 0 \text{ for } \theta = \alpha. \tag{2.1}$$

For numerical solution it is more convenient to reduce (1.13) to a set of two first order equations

$$\psi' = s/6\sqrt{3},\ s' = 18\sqrt{3}\,\psi + \frac{4\sqrt{3}\,\nu\psi s}{\sqrt[3]{s\omega^3+\sqrt{s^2\omega^6+\nu^6\psi^6}}+\sqrt[3]{s\omega^3-\sqrt{s^2\omega^6+\nu^6\psi^6}}} \tag{2.2}$$

with boundary conditions (2.1).

In order to establish the relationship between P(t) and f(t) we consider equilibrium for a sector notionally separated from the wedge with arbitrary radius r

$$P(t) + 2\int_0^\alpha (\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\,r d\theta = 0. \tag{2.3}$$

By substituting (2.3) in an expression for the stress component, after some simplification we obtain

$$P(t) = kJf^{1/3}(t),$$

where

$$J = 2\nu\int_0^\alpha \psi\sin\theta d\theta + \frac{4}{\sqrt{3}}\int_0^\alpha\left[\sqrt[3]{s\omega^3+\sqrt{s^2\omega^6+\nu^6\psi^6}}+\sqrt[3]{\omega^3 s-\sqrt{\nu^6\psi^6+s^2\omega^6}}\right]\cos\theta d\theta.$$

Finally, stress and displacement equations are presented as

$$\begin{gathered}\sigma_r = -\frac{2}{\sqrt{3}}\frac{P(t)}{Jr}\left[\sqrt[3]{\omega^3 s+\sqrt{s^2\omega^6+\nu^6\psi^6}}+\sqrt[3]{\omega^3 s-\sqrt{s^2\omega^6+\nu^6\psi^6}}\right],\\ \tau_{r\theta} = \frac{\nu P(t)}{Jr}\psi,\ \sigma_\theta = 0,\\ u = \frac{P^3(t)}{6\sqrt{3}\,k^3J^3}\frac{s}{r^2},\ v = \frac{P^3(t)}{k^3J^3}\frac{\psi}{r^2}.\end{gathered} \tag{2.4}$$

With P(t) = const there is static deformation. Then arbitrary parameter $\nu = 0$. The solution of Eq. (1.13) will be

$$\psi(\theta) = c_1\,\mathrm{sh}\sqrt{3}\theta + c_2\,\mathrm{ch}\sqrt{3}\theta$$

(c_1 and c_2 are arbitrary constants). From the condition $\psi(0) = 0$ it follows that $c_2 = 0$. Equations for the stress and displacement components have the form

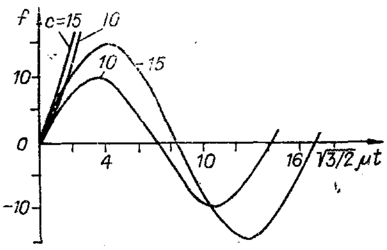

Fig. 1

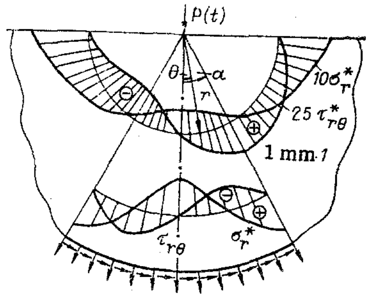

Fig. 2

$$\sigma_r = -\frac{4\sqrt{3}}{\sqrt[3]{6}\,r} c_1^{1/3} k\omega(\theta)\,\mathrm{ch}^{1/3}\sqrt{3}\,\theta,\ \sigma_\theta = \tau_{r\theta} = 0,$$
$$u = \frac{\sqrt{3}\,c_1\,\mathrm{ch}\sqrt{3}\,\theta}{r^2},\ v = \frac{c_1\,\mathrm{sh}\sqrt{3}\,\theta}{r^2}. \tag{2.5}$$

In order to determine unknown constant c_1 we consider static equilibrium for a notionally separated sector of the wedge with arbitrary radius r

$$P + 2\int_0^\alpha \sigma_r \cos\theta \cdot r d\theta = 0.$$

By substituting here an expression for σ_r from (2.5), we have

$$c_1 = \frac{2P^3}{\sqrt{3}\,J_1^3},\ J_1 = 8\int_0^\alpha \omega(\theta)\,\mathrm{ch}^{1/3}\sqrt{3}\,\theta\cos\theta d\theta.$$

Static compression of a homogeneous wedge by a concentrated force applied at the tip has been studied in [8, 9] for the general case of gradual hardening. On the basis of numerical solution of boundary problem (2.1), (2.2) on an ES-1022 computer by the adjustment method [10], curves have been plotted for relative stresses $\sigma_{ij}^* = [Jr/P(t)]\sigma_{ij}$ from Eq. (2.4) (with $\alpha = \pi/6$, $\pi/2$, $\nu = 60$) for the case of inhomogeneity $\omega(\theta) = \exp(\theta^2/2)$ (Fig. 2).

The problem is reduced to a Cauchy problem if it is possible to obtain a sufficient number of boundary conditions at one of the ends of the integration cut-off. It is assumed that it is possible to estimate those values of the functions sought which are not prescribed at the given end of the integration cut-off, when existing information is sufficient for integrating the set of differential equations both in the direction of an increasing argument and in the reverse direction. If the estimates used are correct, then the two solutions found in this way will coincide at internal points of the integration cut-off. Therefore, the problem consists of stagewise improvement of the initial estimates for unknown boundary conditions until the solutions coincide.

Comparison of the results obtained with those for a homogeneous material indicate that inhomogeneity markedly affects the stress-strained state. In fact, with an opening angle $\alpha = \pi/6$ consideration of inhomogeneity leads to an increase in relative normal and tangential stresses by 20 and 10%, and with $\alpha = \pi/2$ it is a factor of 3 and 50% respectively.

3. Now we consider the case when an infinite wedge is bent by a concentrated force P(t) applied to the tip perpendicular to the axis.

Stress and displacement components in this antisymmetrical case are

$$\sigma_r = -\frac{8kf^{1/3}(t)}{r}\frac{\psi'(\theta)\,\omega(\theta)}{\sqrt[3]{[\psi''(\theta) - 3\psi(\theta)]^2 + 16\psi'^2(\theta)}},$$
$$\tau_{r\theta} = \frac{\nu k f^{1/3}(t)}{r}\psi(\theta),\ \sigma_\theta = 0,$$
$$u = \frac{f(t)}{r^2}\psi'(\theta),\ v = \frac{f(t)}{r^2}\psi(\theta),\ \nu = \pm\mu^2\rho/k,$$

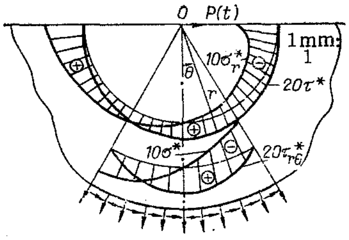

Fig. 3

where function $\psi(\theta)$ is determined from differential Eq. (1.13) with boundary conditions

$$\psi'(\theta) = 0 \text{ for } \theta = 0, \quad \psi(\theta) = 0 \text{ for } \theta = \alpha, \tag{3.1}$$

and function f(t) is determined from quadrature (1.10).

In order to establish the relationship between P(t) and f(t) we consider the equilibrium condition for a sector with arbitrary radius r with a center at the tip notionally separated from the wedge

$$P(t) + 2\int_0^\alpha (\sigma_r \sin\theta + \tau_{\theta r}\cos\theta)\, r d\theta = 0.$$

By substituting here stress components using arrangement (2.2), we find that

$$P(t) = kJf^{1/3}(t),\ J = -2\nu\int_0^\alpha \psi \sin\theta d\theta +$$

$$+\frac{4}{\sqrt{3}}\int_0^\alpha \left[\sqrt[3]{\omega^3 s + \sqrt{s^2\omega^6 + \nu^6\psi^6}} + \sqrt[3]{\omega^3 s - \sqrt{s^2\omega^6 + \nu^6\psi^6}}\right]\sin\theta d\theta.$$

Final stress and displacement equations are presented in the form (2.4). With P(t) = const there is static deformation. Then for unknown function $\psi(\theta)$ we obtain $\psi(\theta) = c_2 \cosh\sqrt{3}\theta$.

Equations for stress and displacement components take the form

$$\sigma_r = -\frac{4\sqrt{3}}{\sqrt[3]{6}\, r} c_2^{1/3} k\omega(\theta)\, \mathrm{sh}^{1/3}\sqrt{3}\theta,\ \sigma_\theta = \tau_{r\theta} = 0,$$

$$u = \frac{\sqrt{3}\, c_2\, \mathrm{sh}\sqrt{3}\theta}{r^2},\ v = \frac{c_2\, \mathrm{ch}\sqrt{3}\theta}{r^2}.$$

In order to determine unknown constant c_2 we consider static equilibrium for a sector with arbitrary radius r notionally separated from the wedge

$$P + 2\int_0^\alpha \sigma_r \sin\theta \cdot r d\theta = 0.$$

Whence

$$c_2 = \frac{2P^3}{\sqrt{3}\, J_2^3},\ J_2 = 8\int_0^\alpha \omega(\theta)\, \mathrm{sh}^{1/3}\sqrt{3}\theta \sin\theta d\theta.$$

On the basis of numerical solution of the boundary problem (2.2), (3.1) on an ES-1022 computer by the adjustment method [10] curves were plotted for relative stresses $\sigma^*_{ij} = [Jr/P(t)]\sigma_{ij}$ from Eq. (2.4) (with $\alpha = \pi/6$, $\pi/2$, $\nu = 60$) for the case of inhomogeneity $\omega(\theta) = \exp(\theta^2/2)$ (Fig. 3).

Comparison of the results obtained with those for a homogeneous material indicate that inhomogeneity leads to an increase in relative stresses. With $\alpha = \pi/6$ relative normal stresses increase by 15%, and with $\alpha = \pi/2$ they increase by a factor of three.

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ANALYSIS AND DESIGN OF STRUCTURAL ELEMENTS WITH OPTIMAL LONGEVITY

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Practically all investigations devoted to the optimal design of structural elements are executed under the assumption of steady creep and do not take into account the circumstance that the cumulative damage process accompanied by a continuous redistribution of the stress therein precedes fracture of the material. The solution of optimization problems with the traditional optimality criterion of the equal strength type results in unrealizable designs in the majority of cases.

In this connection, a variational formulation of the problem of analyzing and designing structural elements with optimal longevity is presented below. It is proposed here to use an optimality criterion that takes account of the total damage over the volume of the material during creep as the target functional. A method is developed for solving this problem on the basis of nonlinear programming methods.

Let a body of volume V bounded by a surface S be loaded by surface loads that are constant in time. The system of equations describing creep of the material and simultaneously taking account of the cumulative damage therein has the form [1]

$$\dot{p}_{ij} = \frac{\Phi_1}{(1-\omega)^m}\frac{s_{ij}}{2S_2}, \quad i, j = 1, 2, 3; \tag{1}$$

$$\dot{\omega} = \Phi_2/(1-\omega)^m, \tag{2}$$

where Φ_1, Φ_2 are homogeneous functions in the stress of degree (n + 1) and (g + 1), $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$; σ_{ij} are stress tensor components, $S_2 = s_{ij}s_{ij}/2$, p_{ij} are creep strain tensor components, ω is the damageability parameter, and m, n, g are material characteristics, and the dot denotes differentiation with respect to the time.

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